

# Entire unbounded constant mean curvature Killing graphs

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## Abstract

In this paper, we provided conditions for an entire constant mean curvature Killing graph lying inside a possible unbounded region to be necessarily a slice.

Let  $N^{n+1}$  denote a complete Riemannian manifold carrying a warped product structure  $N^{n+1} = M^n \times_\rho \mathbb{R}$  with warping function  $\rho \in C^\infty(M)$  such that  $M^n$  is noncompact. Let  $s \in \mathbb{R}$  parametrize the line factor in  $N^{n+1}$ . Then  $Y = \partial/\partial s$  is a Killing vector field free of singularities with integrable orthogonal distribution and  $\rho = |Y|$ . Moreover, the leaves  $M^n \times \{s_0\}$ ,  $s_0 \in \mathbb{R}$ , of the orthogonal distribution to  $Y$  form a foliation by complete isometric totally geodesic hypersurfaces called *slices*.

Fix a slice  $M^n = M^n \times \{0\}$  and denote by  $\Psi: \mathbb{R} \times M^n \rightarrow N^{n+1}$  the flux generated by  $Y$ . The *Killing graph*  $\Sigma(u)$  associated to a function  $u \in C^2(M)$  is the hypersurface

$$\Sigma(u) = \{\Psi(u(x), x) : x \in M^n\}.$$

Recently we proved in [4] that under certain conditions any entire bounded Killing graph with constant mean curvature must be a slice. Bounded means that the graph lies inside a slab, i.e., a set  $M^n \times \mathbb{I}$  where  $\mathbb{I} \subset \mathbb{R}$  is a closed interval.

It is a natural question if the condition that the graph is bounded can be weakened. In fact, this was recently shown to be the case by Rosenberg, Schulze and Spruck [9] for minimal graphs in a Riemannian product in  $M^n \times \mathbb{R}$  where only a lower bound was required. In turn, their result extended a well known theorem due to Bombieri, De Giorgi and Miranda [1] for hypersurfaces in the Euclidean space that also holds for constant mean curvature due to independent work of Chern [2] and Flanders [5].

We denote by  $d(x) = \text{dist}(x, p)$  the geodesic distance from  $p \in M^n$  and by  $B_R(p)$  the geodesic ball centered at  $p$  with radius  $R > 0$ . We also will make use of the notations  $\rho_0 = \sup_M \rho$  and  $\rho_1 = \sup_M |\nabla \rho|$ .

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\*Partially supported by CNPq and FUNCAP.

**Theorem 1.** *Let  $N^{n+1} = M^n \times_\rho \mathbb{R}$  be a Riemannian warped product manifold where  $M^n$  is complete, noncompact such that  $K_M \geq -K_0$  for  $K_0 \geq 0$  and  $\text{Ric}_M \geq 0$ . Assume  $\inf_M \rho > 0$  and  $\|\rho\|_{C^2(M)} < \infty$ . Then, any Killing graph with constant mean curvature  $H \leq 0$  lying inside a region of the form*

$$\mathcal{R} = \left\{ \Psi(s, x) : 0 \leq s \leq \frac{1}{\alpha\beta}(C - \log \alpha\rho(x)) \right\}$$

*for constants  $\alpha > 1$ ,  $C > \log \alpha\rho_0$  and  $\beta \geq n|H|\rho_0 + 2\rho_1$  positive must be a slice.*

From the above theorem one can easily recover the result in [4] by a slight addition to the proof; see Remark 1 for details.

The proof of Theorem 1 is done assuming  $\|\rho\|_{C^1(M)} < \infty$  and  $\text{Ric}_N \geq -L$  for some constant  $L \geq 0$ . That these conditions are weaker than  $\|\rho\|_{C^2(M)} < \infty$  follows from the relation between the Ricci curvatures of  $N^{n+1}$  and  $M^n$  given by

$$\text{Ric}_N(\mathbf{v}, \mathbf{v}) = \text{Ric}_M(\pi_*\mathbf{v}, \pi_*\mathbf{v}) - \frac{1}{\rho} \text{Hess}^M \rho(\pi_*\mathbf{v}, \pi_*\mathbf{v}) - \frac{1}{\rho^3} \langle \mathbf{v}, Y \rangle^2 \Delta^M \rho$$

where  $\pi$  denotes the projection from  $N^{n+1}$  to the factor  $M^n$ .

The region  $\mathcal{R}$  in the above result is determined by the warping function  $\rho$  and may be unbounded as shown in the following example.

**Example 1.** *Let  $M^n$  be a complete noncompact manifold with a pole at  $p \in M^n$  and denote the radial coordinate by  $r$ . Take  $\rho(r) = c + e^{-\psi(r)}$  for some constant  $c > 0$  and a smooth function  $\psi > 0$  with  $\psi' > 0$  such that  $\psi(r) \rightarrow +\infty$  and  $\psi'(r) \rightarrow 0$  as  $r \rightarrow +\infty$ . In this situation, the region  $\mathcal{R}$  is unbounded.*

The main achievement in this paper is a strong improvement of the gradient estimate for Killing graphs when compared with the one in [4]. Here as there, we make use of the Korevaar-Simon method [7] to show the existence of an a priori gradient estimate for a nonnegative solution of the corresponding PDE for constant mean curvature  $H$  over a geodesic ball  $B_R(p)$ .

The function  $G \in C^\infty([0, +\infty))$  in the following result verifies that  $f'' = Gf$  where  $f \in C^\infty([0, +\infty))$  satisfies  $f(0) = 0$ ,  $f'(0) = 1$ , that  $f > 0$  outside the origin and  $f' \geq 0$ . We also use the notation  $\varrho_0 = \sup_{B_R(p)} \rho$  and  $\varrho_1 = \sup_{B_R(p)} |\nabla \rho|$ .

**Theorem 2.** *Let  $N^{n+1} = M^n \times_\rho \mathbb{R}$  be a complete Riemannian manifold satisfying that  $\text{Ric}_N \geq -L$  for some constant  $L \geq 0$ . Assume that the radial sectional curvatures on  $M^n$  along the geodesics issuing from a fixed point  $p \in M^n$  satisfy  $K_{\text{rad}}^M \geq -G(d)$ . Let  $\Sigma(u)$  be a Killing graph with constant mean curvature  $H$  over  $B_R(p)$  such that*

$$0 \leq u \leq \frac{1}{\alpha\beta}(C - \log \alpha\rho) \tag{1}$$

for constants  $\alpha > 1$ ,  $C > \log \alpha \varrho_0$  and  $\beta \geq n|H|\varrho_0 + 2\varrho_1$  positive. Then, we have that

$$|\nabla^M u(p)| \leq D$$

where the constant  $D = D(\alpha, C, \beta, |H|, u(p))$  is given by (20).

*Proof:* It was shown in [3] that a Killing graph  $\Sigma(u)$  has mean curvature  $H$  if and only if the function  $u \in C^2(M)$  satisfies the elliptic PDE of divergence form

$$\operatorname{div}_M \left( \frac{\nabla^M u}{W} \right) - \frac{1}{2\gamma W} \langle \nabla^M \gamma, \nabla^M u \rangle = nH \quad \text{where } \gamma = 1/\rho^2.$$

Here  $H$  is computed with respect to the orientation of the Gauss map given by

$$\mathcal{N} = \frac{1}{W}(\gamma Y - \Psi_* \nabla^M u) \quad \text{where } W = \sqrt{\gamma + |\nabla^M u|^2}. \quad (2)$$

In the sequel  $\nabla$  and  $\Delta$  denote the gradient and Laplace operator on  $\Sigma(u)$ . If  $H$  is constant it is well-known [6] that

$$\Delta \langle Y, \mathcal{N} \rangle = -(|A|^2 + \operatorname{Ric}_N(\mathcal{N})) \langle Y, \mathcal{N} \rangle$$

where  $|A|$  stands for the norm of the second fundamental form of  $\Sigma(u)$ . Thus, using that  $\langle Y, \mathcal{N} \rangle = 1/W$  we obtain

$$\Delta W - \frac{2}{W} |\nabla W|^2 = (|A|^2 + \operatorname{Ric}_N(\mathcal{N}))W. \quad (3)$$

We consider the functions

$$h(d) = \frac{1}{C_R} \int_0^d f(\tau) d\tau \quad \text{where } C_R = \int_0^R f(\tau) d\tau$$

and

$$\xi(s) = e^C \int_0^s e^{-\alpha\beta\tau} d\tau$$

where  $s$  parametrizes the line factor  $\mathbb{R}$  in  $N^{n+1}$ . Let  $\mathcal{K}^+$  denote the solid half-cylinder  $\Psi(\mathbb{R}^+ \times B_R)$  with  $B_R = B_R(p)$  and let  $\phi: \mathcal{K}^+ \rightarrow \mathbb{R}$  be the function

$$\phi(y) = (1 - h(d(x)) - C_0 \xi(s))^+ \quad \text{if } y = \Psi(s, x) \in \mathcal{K}^+$$

where  $C_0 = 1/2 \xi(u(p)) > 0$  and  $+$  means taking the positive part. Let  $\eta$  be the function supported in the portion of the graph  $\Sigma(u)$  over  $B_R$  given by

$$\eta(y) = e^{K\phi(y)} - 1, \quad y = \Psi(u(x), x),$$

for some constant  $K > 0$  and set  $U = \eta W$ .

Let  $\Psi(u(q), q)$  with  $q \in B_R$  be a (necessarily interior) point of maximum of  $U$ . We first consider the case when  $q \in B_R \setminus C(p)$ , where  $C(p)$  is the cut locus of  $p$  in  $M^n$ , thus  $U$  is smooth near  $q$ . Without further reference in the sequel we compute at  $\Psi(u(q), q)$ . It holds that

$$\nabla U = \eta \nabla W + W \nabla \eta = 0$$

and

$$\Delta U = W \Delta \eta + \left( \Delta W - \frac{2}{W} |\nabla W|^2 \right) \eta.$$

Since the Hessian form of  $U$  is nonpositive, we have from (3) that

$$\Delta U = W(\Delta \eta + (|A|^2 + \text{Ric}_N(\mathcal{N}))\eta) \leq 0.$$

Making use of the Ricci curvature assumption, we obtain

$$\Delta \eta \leq L \eta$$

or, equivalently, that

$$\Delta \phi + K |\nabla \phi|^2 \leq \frac{L}{K}. \quad (4)$$

In the sequel, we estimate both terms in the left hand side of (4). We have  $\bar{\nabla} s = \gamma Y$  where  $\bar{\nabla}$  denotes the gradient in  $N^{n+1}$  but also will stand for the Riemannian connection in  $N^{n+1}$ . Where the function  $\phi$  is differentiable and positive, we have

$$\begin{aligned} \bar{\nabla} \phi(y) &= -h'(d) \bar{\nabla} d(y) - C_0 \dot{\xi} \gamma Y(y) \\ &= -h'(d) \Psi_* \nabla^M d(x) - C_0 \dot{\xi} \gamma Y(y) \end{aligned} \quad (5)$$

where  $h' = h'(d)$  and  $\cdot$  indicates the derivative with respect to  $s$ . Then,

$$\langle \bar{\nabla} \phi, \mathcal{N} \rangle = \frac{h'}{W} \langle \nabla^M d, \nabla^M u \rangle - \frac{\gamma}{W} C_0 \dot{\xi}. \quad (6)$$

Therefore,

$$\begin{aligned} |\nabla \phi|^2 &= |\bar{\nabla} \phi|^2 - \langle \bar{\nabla} \phi, \mathcal{N} \rangle^2 \\ &= h'^2 \left( 1 - \frac{1}{W^2} \langle \nabla^M d, \nabla^M u \rangle^2 \right) + \frac{\gamma C_0^2 \dot{\xi}^2}{W^2} |\nabla^M u|^2 + \frac{2\gamma h' C_0 \dot{\xi}}{W^2} \langle \nabla^M d, \nabla^M u \rangle \\ &\geq h'^2 \left( 1 - \frac{1}{W^2} |\nabla^M u|^2 \right) + \frac{\gamma C_0^2 \dot{\xi}^2}{W^2} |\nabla^M u|^2 - \frac{2\gamma h' C_0 \dot{\xi}}{W^2} |\nabla^M u| \\ &= \gamma \dot{\xi}^2 \left( \frac{C_0}{W} |\nabla^M u| - \frac{h'}{\dot{\xi} W} \right)^2. \end{aligned}$$

It follows from the assumption (1) that  $\dot{\xi}(s) \geq \alpha\rho(x)$ , i.e.,  $\sqrt{\gamma}\dot{\xi} \geq \alpha$ . Hence,

$$\left(\frac{C_0}{W}|\nabla^M u| - \frac{h'}{\dot{\xi}W}\right)^2 \geq \frac{C_0^2}{\gamma\dot{\xi}^2}$$

is implied by

$$\left(\frac{C_0}{W}|\nabla^M u| - \frac{h'}{\dot{\xi}W}\right)^2 \geq \frac{C_0^2}{\alpha^2}$$

that, in turn, is equivalent to

$$\left(\frac{C_0}{W}|\nabla^M u| - \frac{h'}{\dot{\xi}W} - \frac{C_0}{\alpha}\right) \left(\frac{C_0}{W}|\nabla^M u| - \frac{h'}{\dot{\xi}W} + \frac{C_0}{\alpha}\right) \geq 0.$$

Clearly, the latter holds if the first factor is nonnegative or, equivalently, if

$$|\nabla^M u| - \frac{1}{\alpha}W \geq \frac{h'}{\dot{\xi}C_0}. \quad (7)$$

Assume that

$$\frac{|\nabla^M u|^2}{\gamma} \geq \frac{1}{\alpha^2 - 1}. \quad (8)$$

It follows that

$$|\nabla^M u| - \frac{1}{\alpha}W \geq 0$$

and thus (7) is implied by

$$|\nabla^M u|^2 - \frac{2}{\alpha}|\nabla^M u|W + \frac{1}{\alpha^2}W^2 \geq \frac{h'^2}{\dot{\xi}^2 C_0^2}. \quad (9)$$

Using that

$$2|\nabla^M u|W \leq |\nabla^M u|^2 + W^2$$

it follows that (9) is implied by

$$|\nabla^M u|^2 - \frac{1}{\alpha}(|\nabla^M u|^2 + W^2) + \frac{1}{\alpha^2}W^2 \geq \frac{h'^2}{\dot{\xi}^2 C_0^2}.$$

But this is equivalent to

$$\left(1 - \frac{1}{\alpha}\right)^2 \frac{|\nabla^M u|^2}{\gamma} - \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \geq \frac{h'^2}{\gamma\dot{\xi}^2 C_0^2}$$

that is implied by

$$\left(1 - \frac{1}{\alpha}\right)^2 \frac{|\nabla^M u|^2}{\gamma} - \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \geq \frac{h'^2}{\alpha^2 C_0^2}$$

Now assume that

$$\frac{|\nabla^M u|^2}{\gamma} \geq \frac{1}{\alpha - 1} + \frac{h'^2}{(\alpha - 1)^2 C_0^2}. \quad (10)$$

Since (10) implies (8), we conclude that the estimate

$$|\nabla \phi|^2 \geq C_0^2. \quad (11)$$

for the first term in (4) holds under (10).

We now estimate the second term in (4). Taking a local orthonormal tangent frame  $\{e_i\}_{i=1}^n$  in  $\Sigma(u)$ , we have

$$\begin{aligned} \Delta \phi &= \sum_{i=1}^n \langle \nabla_{e_i} \nabla \phi, e_i \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i} (\bar{\nabla} \phi - \langle \bar{\nabla} \phi, \mathcal{N} \rangle \mathcal{N}), e_i \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla} \phi, e_i \rangle + \langle \bar{\nabla} \phi, \mathcal{N} \rangle nH. \end{aligned} \quad (12)$$

We obtain from (5) that

$$\langle \bar{\nabla}_{e_i} \bar{\nabla} \phi, e_i \rangle = -h' \langle \bar{\nabla}_{e_i} \bar{\nabla} d, e_i \rangle - h'' \langle \bar{\nabla} d, e_i \rangle^2 - C_0 \langle \bar{\nabla}_{e_i} \dot{\bar{\nabla}} s, e_i \rangle. \quad (13)$$

On the other hand,

$$\langle \bar{\nabla}_{e_i} \bar{\nabla} d, e_i \rangle = \langle \nabla_{\hat{e}_i}^M \nabla^M d, \hat{e}_i \rangle + \gamma^2 \langle Y, e_i \rangle^2 \langle \bar{\nabla}_Y \bar{\nabla} d, Y \rangle$$

where  $e_i = \hat{e}_i + \langle Y, e_i \rangle \gamma Y$ . From our assumptions and the comparison theorem for the Hessian (see Theorem 2.3 in [8]) we obtain

$$\nabla^M \nabla^M d \leq g(d) (\langle \cdot, \cdot \rangle - dd \otimes dd)$$

where  $g = f'/f$ . And since the metric is a warped product, we have

$$\bar{\nabla}_Y \bar{\nabla} d = \frac{1}{\rho} \langle \nabla^M d, \nabla^M \rho \rangle Y.$$

It follows that

$$\langle \bar{\nabla}_{e_i} \bar{\nabla} d, e_i \rangle \leq g(\langle \hat{e}_i, \hat{e}_i \rangle - \langle \nabla^M d, \hat{e}_i \rangle^2) + \frac{1}{2} \langle Y, e_i \rangle^2 |\nabla^M \gamma|.$$

Since  $h' \geq 0$ , we have from (13) that

$$\begin{aligned} \langle \bar{\nabla}_{e_i} \bar{\nabla} \phi, e_i \rangle &\geq -h' (g(1 - \gamma \langle Y, e_i \rangle^2 - \langle \nabla^M d, \hat{e}_i \rangle^2) + \frac{1}{2} \langle Y, e_i \rangle^2 |\nabla^M \gamma|) \\ &\quad - h'' \langle \nabla^M d, \hat{e}_i \rangle^2 - C_0 \langle \bar{\nabla}_{e_i} \dot{\bar{\nabla}} s, e_i \rangle. \end{aligned}$$

On one hand,

$$\begin{aligned}\sum_{i=1}^n \langle \bar{\nabla}_{e_i} \dot{\xi} \bar{\nabla} s, e_i \rangle &= \sum_{i=1}^n \dot{\xi} \langle \bar{\nabla}_{e_i} \bar{\nabla} s, e_i \rangle + \ddot{\xi} \sum_{i=1}^n \langle \bar{\nabla} s, e_i \rangle^2 \\ &= \dot{\xi} (\Delta s - \langle \bar{\nabla} s, \mathcal{N} \rangle nH) + \ddot{\xi} |\nabla s|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta s &= \sum_{i=1}^n \langle \nabla_{e_i} \gamma Y^T, e_i \rangle \\ &= \langle \nabla \gamma, Y^T \rangle + \gamma \sum_{i=1}^n \langle \bar{\nabla}_{e_i} Y, e_i \rangle + nH \langle \gamma Y, \mathcal{N} \rangle \\ &= \langle \bar{\nabla} \gamma, Y^T \rangle + \langle \bar{\nabla} s, \mathcal{N} \rangle nH\end{aligned}$$

where  $Y^T$  denotes the component of  $Y$  tangent to  $\Sigma(u)$ . Hence,

$$\sum_{i=1}^n \langle \bar{\nabla}_{e_i} \dot{\xi} \bar{\nabla} s, e_i \rangle = \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle + \ddot{\xi} |\nabla s|^2.$$

Using that  $h'' = h'g$ , we obtain

$$\sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla} \phi, e_i \rangle \geq -h'(g(n - \gamma|Y^T|^2) + (1/2)|\nabla^M \gamma||Y^T|^2) - C_0 \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle - C_0 \ddot{\xi} |\gamma Y^T|^2.$$

We have from (2) and (6) that

$$\langle \bar{\nabla} \phi, \mathcal{N} \rangle = -h' \langle \bar{\nabla} d, \mathcal{N} \rangle - \frac{\gamma}{W} C_0 \dot{\xi}.$$

Setting

$$\kappa = \frac{1}{\rho} |\nabla^M \rho| = \frac{1}{2\gamma} |\nabla^M \gamma|$$

we obtain from (12) that

$$\Delta \phi \geq -h'(ng + \kappa \gamma |Y^T|^2 + n|H|) - \frac{\gamma}{W} C_0 \dot{\xi} nH - C_0 \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle - C_0 \ddot{\xi} |\gamma Y^T|^2.$$

Hence,

$$\Delta \phi \geq -\frac{n}{C_R} f'(d) - (\kappa \gamma |Y^T|^2 + n|H|) \frac{1}{C_R} f(d) - C_0 \left( \frac{\gamma}{W} \dot{\xi} nH + \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle + \ddot{\xi} |\gamma Y^T|^2 \right).$$

Since  $Y^T = Y - (1/W)\mathcal{N}$ , we have

$$|\gamma Y^T|^2 = \frac{\gamma}{W^2} |\nabla^M u|^2. \quad (14)$$

Thus,

$$\frac{\gamma}{W} = |\gamma Y^T| \frac{\sqrt{\gamma}}{|\nabla^M u|}.$$

Assume that

$$|\nabla^M u|^2 \geq \gamma. \quad (15)$$

Since  $\dot{\xi} \geq 0$  and

$$\langle \bar{\nabla} \gamma, Y^T \rangle \leq 2\kappa |\gamma Y^T|,$$

then we have using (15) that

$$\frac{\gamma}{W} \dot{\xi} n H + \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle + \ddot{\xi} |\gamma Y^T|^2 \leq |\gamma Y^T| (\dot{\xi} (n|H| + 2\kappa) + \ddot{\xi} |\gamma Y^T|).$$

From (8) and (14) we have

$$|\gamma Y^T| \geq \frac{1}{\alpha} \sqrt{\gamma}.$$

Considering that  $\ddot{\xi} = -\alpha\beta\dot{\xi} \leq 0$  we obtain that

$$\frac{\gamma}{W} \dot{\xi} n H + \dot{\xi} \langle \bar{\nabla} \gamma, Y^T \rangle + \ddot{\xi} |\gamma Y^T|^2 \leq |\gamma Y^T| \sqrt{\gamma} \dot{\xi} (n|H| \rho + 2|\nabla^M \rho| - \beta).$$

Now the choice of  $\beta$  gives that

$$\Delta \phi \geq -\frac{n}{C_R} f'(d) - (\kappa \gamma |Y^T|^2 + n|H|) \frac{1}{C_R} f(d).$$

We have from (14) that  $\gamma |Y^T| \leq 1$  and from  $f' \geq 0$  that  $f(d) \leq f(R)$ . Hence,

$$\Delta \phi \geq -n \frac{f'(d)}{C_R} - (\kappa + n|H|) \frac{f(R)}{C_R}.$$

It follows that

$$\Delta \phi > -2A \quad (16)$$

where

$$2A = \frac{n}{C_R} \sup_{B_R} f'(d) + \frac{f(R)}{C_R} (n|H| + \sup_{B_R} \kappa).$$

From (4), (11) and (16) we obtain

$$\frac{L}{K} \geq \Delta \phi + K |\nabla \phi|^2 > -2A + K C_0^2. \quad (17)$$

Taking

$$K > \frac{1}{C_0^2} \left( A + \sqrt{A^2 + C_0^2 L} \right), \quad (18)$$



we obtain a contradiction in (17). Thus, we conclude from (10) and (15) that

$$\frac{|\nabla^M u|^2}{\gamma}(q) < D_0 = \max_{B_R} \left\{ 1, \frac{1}{\alpha - 1} + \frac{4f^2(R)\xi^2(u(p))}{(\alpha - 1)^2 C_R^2} \right\}. \quad (19)$$

Then,

$$W(q) \leq D_1 = \frac{1}{r_0} \sqrt{1 + D_0}$$

where  $r_0 = \min_{B_R} \rho$ . Hence,

$$U(p) = (e^{K/2} - 1)W(p) \leq U(q) \leq D_1(e^K - 1)$$

where  $K$  is given by (18). It follows that

$$|\nabla^M u(p)| \leq D = D_1(e^{K/2} + 1). \quad (20)$$

To conclude the proof we observe that the same argument given in [9] proves that the restriction that  $q \notin C(p)$  can be dropped.  $\square$

*Proof of Theorem 1:* We claim that there is a global gradient estimate obtained by taking  $R \rightarrow +\infty$  in Theorem 2. To see this we choose the function

$$f(t) = \frac{1}{\sqrt{K_0}} \sinh \sqrt{K_0} t$$

that satisfies all the requirements in that result. Then, we have  $G = K_0$  and

$$C_R = \frac{1}{K_0} \left( \cosh \sqrt{K_0} R - 1 \right).$$

Since  $f(R)/C(R) \rightarrow 1$  as  $R \rightarrow +\infty$ , it follows from (19) that we can take

$$D_0 = \max \left\{ 1, \frac{1}{\alpha - 1} + \frac{4K_0 \xi^2(u(p))}{(\alpha - 1)^2} \right\}.$$

We also have that

$$2A \leq 2A_1 = nK_0 + (n|H| + \rho_1/r_0)\sqrt{K_0}$$

where  $r_0 = \inf_M \rho$ . Since  $0 \leq \xi < e^C/\alpha\beta$ , then

$$C_0 \geq C_1 = \alpha\beta/2e^C.$$

Thus, from (18) we can take

$$K > \frac{1}{C_1^2} \left( A_1 + \sqrt{A_1^2 + C_1^2 L} \right),$$

and the claim follows from (20).

We now make use of part of the argument in [4] to prove that  $H = 0$ . It was shown there the Omori-Yau maximum principle for the Laplacian holds on  $\Sigma(u)$ . Moreover, we have that  $u = s|_\Sigma$  satisfies

$$\Delta u = \frac{n\gamma}{\sqrt{\gamma + |\nabla^M u|^2}} H + \frac{1}{\gamma} \langle \nabla \gamma, \nabla u \rangle. \quad (21)$$

Being  $u$  a bounded function from below on  $\Sigma(u)$ , the Omori-Yau maximum principle assures that there exists a sequence  $\{y_k\}_{k \in \mathbb{N}}$  such that

$$|\nabla u(y_k)| < 1/k \quad \text{and} \quad \Delta u(y_k) > -1/k.$$

Since we have already shown that the coefficient of  $H$  is bounded from below by a positive number, we obtain from (21) that  $H = 0$ .

**Remark 1.** Notice that the above argument can be used to show that a graph of any constant mean curvature  $H$  inside a slab must be minimal.

We conclude the proof with an argument from [4] or [9]. Now the PDE can be written as  $Lu = 0$  where

$$Lu = e^{-\varphi} \operatorname{div}_\Sigma e^\varphi \nabla u, \quad \varphi = 2 \log \rho.$$

In a system of coordinates  $\{x^i\}_{i=1}^n$  in  $M^n$  with  $\sigma_{ij} = \langle \partial_{x^i}, \partial_{x^j} \rangle$ , we have

$$L = e^{-\varphi} \operatorname{div}_\Sigma (e^\varphi g^{ij} u_i \partial_{x^j})$$

where

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2} \quad \text{and} \quad u^i = \sigma^{ik} u_k.$$

Since there is a global gradient estimate, then  $L$  is an uniformly elliptic operator in divergence form. Hence, if we view  $L$  as a operator acting on  $M^n$  and since  $\operatorname{Ric}_M \geq 0$ , it follows from Theorem 7.4 in [10] that  $u$  is constant.  $\square$

## References

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